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# Finding Disjoint Trees in Planar Graphs in Linear Time

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ABSTRACT. We show that for each fixed $k$ there exists a linear-time algorithm for the problem: *given:* an undirected plane graph $G = (V, E)$ and subsets $X_1, \ldots, X_p$ of $V$ with $|X_1 \cup \cdots \cup X_p| \leq k$; *find:* pairwise vertex-disjoint trees $T_1, \ldots, T_p$ in $G$ such that $T_i$ covers $X_i$ $(i = 1, \ldots, p)$.

## 1. Introduction

Consider the following *disjoint trees problem*:

given: an undirected graph $G = (V, E)$ and subsets $X_1, \ldots, X_p$ of $V$;
find: pairwise vertex-disjoint trees $T_1, \ldots, T_p$ in $G$ such that $T_i$ covers $X_i$ $(i = 1, \ldots, p)$.

(We say that tree $T_i$ *covers* $X_i$ if each vertex in $X_i$ is a vertex of $T_i$.)

Robertson and Seymour [5] gave an algorithm for this problem that runs, for each fixed $k$, in time $O(|V|^3)$ for inputs satisfying $|X_1 \cup \cdots \cup X_p| \leq k$. (Recently, Reed gave an improved version with running time $O(|V|^2 \log |V|)$.) In this paper we show that if we moreover restrict the input graphs to planar graphs there exists a *linear-time* algorithm:

THEOREM. *There exists an algorithm for the disjoint trees problem for planar graphs that runs, for each fixed k, in time $O(|V|)$ for inputs satisfying*

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$|X_1 \cup \cdots \cup X_p| \leq k$.

If we do not fix an upper bound $k$ on $|X_1 \cup \cdots \cup X_p|$, the disjoint trees problem is NP-hard (D.E. Knuth, see [1]), even when we restrict ourselves to planar graphs and each $X_i$ is a pair of vertices (Lynch [2]).

Our result extends a result of Suzuki, Akama, and Nishizeki [7] stating that the disjoint trees problem is solvable in linear time for planar graphs for each fixed upper bound $k$ on $|X_1 \cup \cdots \cup X_p|$, when

(1) there exist two faces $f_1$ and $f_2$ such that each vertex in $X_1 \cup \cdots \cup X_p$ is incident with at least one of $f_1$ and $f_2$.

(In fact, they showed more strongly that the problem (for nonfixed $k$) is solvable in time $O(k|V|)$. Indeed, recently Ripphausen, Wagner, and Weihe [4] showed that it is solvable in time $O(|V|)$.)

Equivalent to a linear-time algorithm for the disjoint trees problem (for fixed $k$) is one for the following "realization problem". Let $G = (V, E)$ be a graph and let $X \subseteq V$. For any $E' \subseteq E$ let $\Pi(E')$ be the partition $\{K \cap X | K$ is a component of the graph $(V, E')$ with $K \cap X \neq \emptyset\}$ of $X$. We say that $E'$ *realizes* $\Pi$ if $\Pi = \Pi(E')$. We call a partition of $X$ *realizable in* $G$ if it is realized by at least one subset $E'$ of $E$. Now the *realization problem* is:

given: a graph $G = (V, E)$ and a subset $X$ of $V$;

find: subsets $E_1, \ldots, E_N$ of $E$ such that each realizable partition of $X$ is realized by at least one of $E_1, \ldots, E_N$.

We give an algorithm for the realization problem for planar graphs that runs, for each fixed $k$, in time $O(|V|)$ for inputs satisfying $|X| \leq k$. In [3] we extend this result to graphs embedded on any fixed compact surface.

## 2. Realizable partitions

We will use the following lemma of Robertson and Seymour [6], saying that any vertex that is "far away" from $X$ and also is not on any "short" curve separating $X$, is irrelevant for the realization problem and can be left out from the graph.

Let $G = (V, E)$ be a plane graph (that is, a graph embedded in the plane $\mathcal{R}^2$). For any curve $C$ on $\mathcal{R}^2$, the *length* length$(C)$ of $C$ is the number of times $C$ meets $G$ (counting multiplicities). We say that a curve $C$ *separates* a subset $X$ of $\mathcal{R}^2$ if $X$ is contained in none of the components of $\mathcal{R}^2 \setminus C$. (So $C$ separates $X$ if $C$ intersects $X$.)

LEMMA. *There exists a computable function $g : \mathcal{N} \longrightarrow \mathcal{N}$ with the following property. Let $G = (V, E)$ be a plane graph, let $X \subseteq V$ and let $v \in V$ be such that each closed curve $C$ traversing $v$ and separating $X$ satisfies* length$(C) \geq g(|X|)$; *then each partition of $X$ realizable in $G$ is also realizable in $G - v$.*

[$G - v$ is the graph obtained from $G$ by deleting $v$ and all edges incident with $v$.]

Moreover, we use the following easy proposition, enabling us to reduce the realization problem to smaller problems.

PROPOSITION 1. *Let $G = (V, E)$ be an undirected graph and let $X \subseteq V$. Moreover, let $V_1, \dots, V_n, Y$ be subsets of $V$ such that*

(2) (i) *each edge of $G$ is contained in at least one of $V_1, \dots, V_n$;*
(ii) *$X \subseteq Y$ and $V_i \cap V_j \subseteq Y$ for each $i, j \in \{1, \dots, t\}$ with $i \neq j$.*

*Let $E_{i,1}, \dots, E_{i,N_i}$ form a solution for the realization problem with input $\langle V_i \rangle, V_i \cap Y$ $(i = 1, \dots, n)$. Then the sets $E_{1,j_1} \cup \dots \cup E_{n,j_n}$, where $j_i$ ranges over $1, \dots, N_i$ (for $i = 1, \dots, n$), form a solution for the realization problem with input $G, X$.*

[$\langle W \rangle$ denotes the subgraph of $G$ induced by $W$.]

## 3. Proof of the theorem

We show that, for each fixed $k$, there exists a linear-time algorithm for the realization problem for plane graphs $G = (V, E)$ and subsets $X$ of $V$ with $|X| \leq k$. We may assume that $G$ is connected.

For any subset $W$ of $V$ let $\delta(W)$ be the set of vertices in $W$ that are adjacent to at least one vertex in $V \setminus W$. Let $W^o := W \setminus \delta(W)$.

Let $H$ be the graph with vertex set $V$, where two vertices $v, v'$ are adjacent if and only if there exists a face of $G$ that is incident with both $v$ and $v'$. For any subset $W$ of $V$, let $\kappa(W)$ denote the number of components of the subgraph of $H$ induced by $W$. Note that $\kappa(W)$ can be computed in linear time.

We say that $W$ is *linked* if $\kappa(W) = 1$. Observe that if $W \neq \emptyset$ then

(3) $W$ is linked if and only if $G$ does not contain a circuit $C$ splitting $W$.

Here we say that $C$ *splits* $W$ if $C$ does not intersect $W$ and $\emptyset \neq W \cap \text{int}C \neq W$, where int$C$ denotes the (open) area of $\mathcal{R}^2$ enclosed by $C$.

We apply induction on $\kappa(X)$. If $\kappa(X) \leq 2$, the problem can be reduced to one satisfying (1). Indeed, if $\kappa(X) = 2$ we can find in linear time a collection $F$ of faces of $G$ such that the subspace $X \cup \bigcup_{f \in F} f$ of $\mathcal{R}^2$ has two connected components and such that $|F| \leq |X|$. Choose two faces $f, f' \in F$ and a vertex $v \in X$ incident with both $f$ and $f'$. "Open" the graph at $v$, by splitting $v$ into two new vertices, joining $f$ and $f'$ to form one new face. After this is repeated $|F| - 3$ times, the faces in $F$ are replaced by two faces $f_1$ and $f_2$ and the vertices in $X$ are split (or not) to a set $X'$ of $|X| + |F| - 2$ vertices, such that each vertex in $X'$ is incident with $f_1$ or $f_2$. By the result of Suzuki, Akama, and Nishizeki [7] we can solve the realization problem for the new graph and $X'$ in linear time. This directly gives a solution for the realization problem for the original realization problem. We proceed similarly if $\kappa(X) = 1$.

If $\kappa(X) > 2$ we proceed as follows. Let $X_1, \dots, X_t$ be the components of the subgraph of $H$ induced by $X$. (So $t = \kappa(X) \leq k$.) We may assume that $\delta(X_i) = X_i$ for each $i = 1, \dots, t$ (by attaching to each vertex in $X_i$ a new vertex of valency 1). Let $p$ be a nonnegative integer. A *p-neighbourhood* is a collection $W_1, \dots, W_t$ of pairwise disjoint linked subsets of $V$ with the following properties:

(4) (i) for $i = 1, \dots, t$, $W_i \supseteq X_i$, and if $W_i \neq X_i$ then $|\delta(W_i)| = p$

(ii) for all distinct $i, j \in \{1, \dots, t\}$, there are $p$ vertex-disjoint paths in $G$ between $W_i$ and $W_j$.

We note:

PROPOSITION 2. *Let $W_1, \dots, W_t$ be a p-neighbourhood. Let $i, j \in \{1, \dots, t\}$ be distinct, and let $T$ be a set of vertices intersecting each path from $W_i$ to $W_j$ such that $|T| = p$. Then $T$ is linked.*

*Proof.* Suppose not. Let $C$ be a circuit in $G$ splitting $T$. Let $U_i$ and $U_j$ be the sets of vertices that can be reached from $W_i$ and $W_j$, respectively, without intersecting $T$. So $U_i \cap U_j = \emptyset$. Then $U_i \cap C = \emptyset$ or $U_j \cap C = \emptyset$, since otherwise all vertices in $C$ belong both to $U_i$ and $U_j$. We may assume that $U_i \cap C = \emptyset$. Hence we may assume moreover that $U_i$ is contained in $\text{int}C$ (as $U_i$ is linked). Then each path from $W_i$ to $W_j$ intersects $T \cap \text{int}C$, contradicting the facts that there exist $p$ disjoint such paths and that $|T \cap \text{int}C| < |T| = p$. ∎

In particular, $\delta(W_i)$ is linked for all $i$. (If $W_i = X_i$ then $\delta(W_i) = \delta(X_i) = X_i$.)

Call a $p$-neighbourhood $W_1, \dots, W_t$ *maximal* if for each $i = 1, \dots, t$ and for each linked $U$ satisfying $W_i \subset U \subseteq V \setminus \bigcup_{j \neq i} W_j$ one has $|\delta(U)| > p$.

First we describe an algorithm which, given a $p$-neighbourhood $W_1, \dots, W_t$, finds a maximal $p$-neighbourhood:

1. Choose $i \in \{1, \dots, t\}$. Determine an inclusionwise maximal set $U$ satisfying $W_i \subseteq U \subseteq V \setminus \bigcup_{j \neq i} W_j$ and $|\delta(U)| = p$. Replace $W_i$ by $U$. If no such $U$ exists, we leave $W_i$ unchanged.

2. Repeat for all $i \in \{1, \dots, t\}$ in turn. This gives a maximal $p$-neighbourhood.

Note that by Proposition 2, $\delta(U)$ in Step 1 is linked, and hence $U$ is linked. Note moreover that Step 1 can be performed in time $O(p|V|)$ with the Ford-Fulkerson augmenting path method (one augmenting path can be found in time $O(|V|)$). See also [4].

Second we give an algorithm which, given a maximal $p$-neighbourhood, finds either a $p+1$-neighbourhood or a reduction for the realization problem:

1. If there exist $i \neq j$ and a vertex $v$ such that both $W_i \cup \{v\}$ and $W_j \cup \{v\}$ are linked, apply Proposition 1 to $V_1 := W_i \cup \{v\}, V_2 := W_j \cup \{v\}, V_3 := V \setminus (W_i^o \cup W_j^o)$ and $Y := X \cup \delta(W_i) \cup \delta(W_j) \cup \{v\}$.

Otherwise, for each $i = 1, \dots, t$ with $|\delta(W_i)| = p$, choose a vertex $v_i \in V \setminus W_i$ such that $W_i \cup \{v_i\}$ is linked, and let $U_i := W_i \cup \{v_i\}$; for all other $i$ let $U_i := W_i$.

2. If there exist $i \neq j$ such that there do not exist $p+1$ disjoint paths connecting $U_i$ and $U_j$, find a subset $U$ of $V$ such that $U_i \subseteq U, U_j \subseteq U' := V \setminus U^o$ and $|\delta(U)| = p$. Apply Proposition 1 to $V_1 := W_1, \ldots, V_t := W_t, V_{t+1} := (U \setminus (W_1^o \cup \cdots \cup W_t^o)) \cup \delta(U), V_{t+2} := (U' \setminus (W_1^o \cup \cdots \cup W_t^o)) \cup \delta(U)$ and $Y := X \cup \delta(W_1) \cup \cdots \cup \delta(W_t) \cup \delta(U)$.

3. Otherwise, $U_1, \ldots, U_t$ form a $p+1$-neighbourhood.

PROPOSITION 3. *In Step 1, if there exist $i$ and $j$ as stated, then $\kappa(V_h \cap Y) < t$ for $h = 1, 2, 3$.*

*Proof.* Without loss of generality, $i = 1$ and $j = 2$. We have $\kappa(V_1 \cap Y) = \kappa(X_1 \cup \delta(W_1) \cup \{v\}) \leq 2 < t$, since both $X_1$ and $\delta(W_1) \cup \{v\}$ are linked. Similarly, $\kappa(V_2 \cap Y) \leq 2 < t$.

Finally, $\kappa(V_3 \cap Y) < t$, since $V_3 \cap Y = X_3 \cup \cdots \cup X_t \cup \delta(W_1) \cup \delta(W_2) \cup \{v\}$, where $X_3, \ldots, X_t$ and $\delta(W_1) \cup \delta(W_2) \cup \{v\}$ are linked (as $\delta(W_1) \cup \{v\}$ and $\delta(W_2) \cup \{v\}$ are linked). ∎

PROPOSITION 4. *Let $A, B \subseteq V$ such that $\delta(A)$ and $\delta(B)$ are linked, and such that $B \not\subseteq A^o$ and $A^o \cup B^o \neq VG$. Then $\delta(A) \cup (A \cap \delta(B))$ is linked.*

*Proof.* Suppose $\delta(A) \cup (A \cap \delta(B))$ is not linked. Let $C$ be a circuit in $G$ splitting $\delta(A) \cup (A \cap \delta(B))$. Since $\delta(A)$ is linked, we may assume that $\delta(A) \subset \text{int}C$. Since $C$ splits $\delta(A) \cup (A \cap \delta(B))$, we know that there are vertices in $A \cap \delta(B)$ that are in the exterior of $C$.

Since $G$ is connected, there exists a path in $G$ from a vertex in $A$ in the exterior of $C$ to a vertex of $C$ disjoint from $\delta(A)$, and hence $C$ intersects $A$. Therefore, $VC \subseteq A$. Hence every vertex of $G$ in the exterior of $C$ belongs to $A$. As $\delta(B)$ is linked and as $\delta(B)$ does not intersect $C$ (because $A \cap \delta(B)$ does not intersect $C$), we have that $\delta(B)$ is contained in the exterior of $C$. As $B \not\subseteq A^o$ this implies that each vertex in $\text{int}C$ is contained in $B$. So $A^o \cup B^o = VG$, contradicting the assumption. ∎

PROPOSITION 5. *In Step 2, if there exist $i$ and $j$ as stated, then $\kappa(V_h \cap Y) < t$ for $h = 1, \ldots, t+2$.*

*Proof.* Without loss of generality, $i = 1$ and $j = 2$. By the maximality of $W_1$ we know that $U$ intersects at least one of $W_2, W_3 \ldots, W_t$. So $U$ intersects at least two of $W_1, \ldots, W_t$. Similarly, $U'$ intersects at least two of $W_1, \ldots, W_t$.

For each $h = 1, \ldots, t$ we have $\kappa(V_h \cap Y) \leq 2 < t$, since $V_h \cap Y = X_h \cup \delta(W_h) \cup (W_h \cap \delta(U))$ and since $\delta(W_h) \cup (W_h \cap \delta(U))$ is linked by Proposition 4. (Note that $U \not\subseteq W_h^o$ since $U$ intersects at least two of $W_1, \ldots, W_t$, and that $U^o \cup W_h^o \neq VG$ since $U'$ intersects at least two of $W_1, \ldots, W_t$.)

Next we show $\kappa(V_{t+1} \cap Y) < t$. Note that $V_{t+1} \cap Y = \delta(U) \cup (U \cap (\delta(W_1) \cup \cdots \cup \delta(W_t)))$. Since $U'$ intersects at least two of $W_1, \ldots, W_t$, it suffices to show that if $U'$ intersects $W_h$ then $\delta(U) \cup (U \cap \delta(W_h))$ is linked.

Suppose $U'$ intersects $W_h$ and $\delta(U) \cup (U \cap \delta(W_h))$ is not linked. As $\delta(U)$ and $\delta(W_h)$ are linked (by Proposition 2), Proposition 4 implies that $W_h \subseteq U^o$ or $W_h^o \cup U^o = VG$. However, $W_h \subseteq U^o$ contradicts the fact that $W_h$ intersects $U'$. Moreover, $W_h^o \cup U^o = VG$ contradicts the fact that there is another $W_{h'}$ intersecting $U'$.

This shows $\kappa(V_{t+1} \cap Y) < t$. Similarly, $\kappa(V_{t+2} \cap Y) < t$. ∎

Finally we give the algorithm which finds a reduction:

> Starting with the 0-neighbourhood $X_1, \ldots, X_t$, for $p = 0, 1, .., 2g(k) - 1$ apply the above algorithms to find a reduction or a $2g(k)$-neighbourhood.
>
> If we find a $2g(k)$-neighbourhood $W_1, \ldots, W_t$, then for all distinct $i, j \in \{1, \ldots, t\}$, find a shortest path $P_{i,j}$ in $H$ between $W_i$ and $W_j$. Among all $P_{i,j}$ choose one, $P := P_{1,2}$ say, of minimum length.
>
> If length$(P) > 2g(k)$, delete from $G$ all vertices of $P$ except the first $g(k)$ and the last $g(k)$. If length$(P) \leq 2g(k)$ leave $G$ unchanged. Call the new graph $G'$.
>
> Let $R$ be the set of vertices in $P$ that are not deleted. Apply Proposition 1 to $G'$ and $V_1 := W_1, V_2 := W_2, V_3 := V \setminus (W_1^o \cup W_2^o)$ and $Y := X \cup \delta(W_1) \cup \delta(W_2) \cup R$.

PROPOSITION 6. *In $G'$, $\kappa(V_h \cap Y) < t$ for $h = 1, 2, 3$.*

*Proof.* $\kappa(V_1 \cap Y) = \kappa(X_1 \cup \delta(W_1)) \leq 2 < t$. Similarly, $\kappa(V_2 \cap Y) < t$. Finally, $\kappa(V_3 \cap Y) = \kappa(X_3 \cup \cdots \cup X_t \cup \delta(W_1) \cup \delta(W_2) \cup R) < t$ since $\delta(W_1) \cup \delta(W_2) \cup R$ is linked. ∎

PROPOSITION 7. *Deleting the vertices does not affect realizability.*

*Proof.* Let $Q$ be the set of vertices deleted. We must show that for any vertex $v \in Q$, any closed curve $C$ traversing $v$ and separating $X$ has at least $g(k)$ intersections with $G - (Q \setminus \{v\})$ (since it means by the lemma that we can delete $v$, even after having deleted all other vertices in $Q$). In other words, any closed curve in $\mathcal{R}^2$ intersecting $Q$ and separating $X$ should have at least $g(k) - 1$ intersections with $G - Q$.

Let $C$ be a closed curve intersecting $Q$ and separating $X$, having a minimum number $p$ of intersections with $G - Q$. We may assume that $C$ intersects $G$ only in vertices of $G$. Suppose $p \leq g(k) - 2$. It is not difficult to see that, by the minimality of $p$, there exist $x, y \in Q$ on $C$ (possibly $x = y$) such that, if we denote by $K$ and $K'$ the two (closed) $x - y$ parts of $C$, then one of these parts, $K$ say, intersects $G$ only in $Q$, while $K'$ intersects $Q$ only in the end points $x$ and $y$ of $K'$. We may assume that $K$ is part of $P$. Hence as $P$ is a shortest path, length$(K) \leq$ length$(K') = p + 2$. So length$(C) =$ length$(K) +$ length$(K') - 2 \leq 2p + 2 \leq 2g(k) - 2$.

Hence $C$ does not intersect any face incident with any point in any $W_i$, since otherwise $C$ would contain a curve of length at most $g(k)-1$ connecting $Q$ and $W_i$, contradicting the minimality of $P$. As $C$ separates $X$, there exist $i \neq j$ such that $W_i$ and $W_j$ are in different components of $\mathcal{R}^2 \setminus C$. This contradicts the facts that there exist $2g(k)$ pairwise disjoint paths from $W_i$ to $W_j$ and that $\text{length}(C) < 2g(k)$. ∎

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